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On expression of eta-product by theta series

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1 Introduction

Let $\eta(\tau)$ be the Dedekind eta function defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

where q $=\exp(2\pi i\tau)$ and τ lies in the complex upper half plane $\mathcal{H}=\{\tau|Im(\tau)>0\}$. Let N be a positive integer, let $e_1, e_2, \cdots, e_N$ are integers, and let f be a meromorphic function over $\mathcal{H}$ of the form

$$\prod_{\substack{i|N \\ i>0}} \eta(i\tau)^{e_i}.$$

If all of the e_i are non-negative, we say that f is an eta-product.
Serre [12] has given the following identities relating certain eta-product and theta series associated to a pair of quadratic forms. (see [12], p260)

Let p be a prime number such that $p \equiv -1 \pmod{24}$. Consider the following pair of primitive binary quadratic forms with discriminant $-p$:

$$Q_1:\ 6x^2+xy+\frac{p+1}{24}y^2, \qquad Q_2:\ 6x^2+5xy+\frac{p+25}{24}y^2.$$

Then we have

$$\frac{1}{2}(\vartheta_{Q_1}(\tau)-\vartheta_{Q_2}(\tau))=\eta(\tau)\eta(p\tau),$$

where $\vartheta_{Q_1}(\tau)$ and $\vartheta_{Q_2}(\tau)$ are the theta series associated to Q_1 and Q_2.

We will extend this relation. Our result is the following.

Theorem 1.1
Let N be a squarefree positive integer such that $N \equiv -1 \pmod{24}$. Consider the following two primitive binary quadratic forms with discriminant $-N$:

$$Q_1 : \ 6x^2 + xy + \frac{N+1}{24}y^2, \qquad Q_2 : \ 6x^2 + 5xy + \frac{N+25}{24}y^2.$$

Then we have

$$\frac{1}{2}(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)) = \eta(\tau)\eta(N\tau).$$

In theorem1.1, we see that $\frac{1}{2}(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau))$ and $\eta(\tau)\eta(N\tau)$ are cusp forms on $\Gamma_0(N)$ of weight 1 and character $\chi_{-N}(*) = (\frac{-N}{*})$ (Jacobi symbol) i.e. $\frac{1}{2}(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)), \eta(\tau)\eta(N\tau) \in \mathcal{S}_1(\Gamma_0(N), \chi_{-N})$. The space of $\mathcal{S}_1(\Gamma_0(N), \chi_{-N})$ is very interesting but difficult. New forms of weight 1 correspond to Galois representations of the $G_{\mathbf{Q}} = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.

We review two-dimensioanal Galois representations and new forms of weight 1. Let p be a prime and $g = \sum_{n=1}^{\infty} a_n q^n$ be a normalised newform on $\Gamma_0(p)$ of weight 1. Then Deligne-Serre's theorem shows that there is an irreducible two-dimensional linear representation ρ of $G_{\mathbf{Q}}$, with Artin conductor p, such that $L(\rho, s) = \sum_{n=1}^{\infty} a_n n^{-s}$, where $L(\rho, s)$ is the Artin L-series. We say that g is of dihedral type (resp. type S_4, type A_5) if ρ is of dihedral type (resp. type S_4, type A_5) i.e. the image of projective linear representation $\widetilde{\rho}(G_{\mathbf{Q}}) \subset PGL_2(\mathbf{C})$ is isomorphic to the dihedral group D_n (resp. S_4, A_5). Now let $-d$ be a discriminant of imaginary quadratic field. For $f \in \mathbf{N}$ let $H(-df^2)$ be the set of equivalence classes of primitive positive definite binary quadratic forms with discriminant $-df^2$. Let $\rho : G_{\mathbf{Q}} \to GL_2(\mathbf{C})$ be an irreducible odd representation with Artin conductor df^2 and such that $\rho(G_{\mathbf{Q}})$ is a generalized dihedral group. Then the fixed field of the kernel of ρ is contained in the ring class field K_f of $K = \mathbf{Q}(\sqrt{-d})$ with conductor f and ρ is determined by a character $\chi : H(-df^2) \to \mathbf{C}^*$, namely $\rho = Ind\,{}_{Gal(K_f/K)}^{Gal(K_f/\mathbf{Q})}(\chi)$. For this ρ, there exists a normalized newform g in $\mathcal{S}_1(\Gamma_0(-df^2), \chi_{-d})$ such that $L(\rho, s) = L(g, s)$. It is given by

$$g = \frac{1}{\omega} \sum_{Q \in H(-df^2)} \chi(Q)\ \vartheta_Q(\tau)$$

(cf. [1]), where ω is a number of roots of unity in $\mathbf{Q}(\sqrt{-d})$.

In section 4 , by using theorem1.1, we see that some $\eta(\tau)\eta(N\tau)$ are linear combination of cusp forms which correspond to representation of dihedral type. Moreover we study the subspace which is generated by $\{\vartheta_Q(\tau) \mid Q \in H(-N)\} \subset \mathcal{S}_1(\Gamma_0(-N), \chi_{-N})$.

2 Preliminaries

In this section, we recall some results about eta-product and theta series associated to quadratic form. First, we review eta-product.

Suppose that $f(\tau) = \prod_{0<i|N} \eta(i\tau)^{e_i}$ is an eta-product which satisfies the following properties :

$$1. \sum_{0<i|N} ie_i \equiv 0 \pmod{24},$$

$$2. \sum_{0<i|N} \frac{N}{i} e_i \equiv 0 \pmod{24}.$$

Then $f(\tau)$ satisfies

$$f(\frac{a\tau+b}{c\tau+d}) = \chi(d)(c\tau+d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $k := \frac{1}{2}\sum_{0<i|N} e_i$, $\chi(d) := (\frac{(-1)^k s}{d})$ (Jacobi symbol), and $s := \prod_{0<i|N} i^{e_i}$. That is, $f(\tau)$ is a weakly modular form of level N, weight k and character χ(see [3], p90).

A complete set of representatives for the cusps of $\Gamma_0(N)$ is

$$\mathcal{C}_N = \{\frac{a}{c} \in \mathbf{Q} \ ; \ c|N, \ 1 \le a \le N \ gcd(a, N) = 1$$

$$and \ \frac{a}{c} = \frac{a'}{c} \ \Leftrightarrow \ a \equiv a' \pmod{gcd(c, \frac{N}{c})}\}.$$

Let $\frac{a}{c} \in \mathcal{C}_N$. Then the order of zero of $f(\tau) = \prod_{0<i|N} \eta(i\tau)^{e_i}$ at $\frac{a}{c}$ is

$$\nu_{\frac{a}{c}} = \frac{h_c}{24} \sum_{0<i|N} \frac{gcd(i,c)^2}{i} e_i \ , \tag{1}$$

where $h_c = \frac{N}{gcd(c^2,N)}$ is the width of the cusp $\frac{a}{c}$(see [6], p49).

Next, we review the theta series associated to quadratic form. Let $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}A^t\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}x_ix_j$ ($\mathbf{x} = (x_1, \ldots, x_r)$, $x_j \in \mathbf{R}$, $A \in M_r(\mathbf{R})$) be a positive defnite integral quadratic form, that is $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, and $a_{ij} = a_{ji}$ is an inetger, a_{ii} is an even inetger, i.e. $A = (a_{ij})$ is an integral symmetric matrix. The theta series associated to Q is defined by

$$\vartheta_Q(\tau) = \sum_{x \in \mathbf{Z}^r} q^{Q(x)}.$$

Assume r is even and put $r = 2k$. The bacis result, due to Schoeneberg, is that $\vartheta_Q(\tau) \in \mathcal{M}_k(N, \chi)$, where N is the least positive integer such that NA^{-1} is even integral and $\chi(d) := (\frac{(-1)^k det A}{d})$ (Jacobi symbol)(see [11] chapter VI).

In theorem1.1 we can write $Q_1(x, y) = \frac{1}{2}(x, y) \begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Since $N \begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix}^{-1}$ is even integral, we have $\vartheta_{Q_1}(\tau) \in \mathcal{M}_1(\Gamma_0(N), \chi_{-N})$. Similarly, we have $\vartheta_{Q_2}(\tau) \in \mathcal{M}_1(\Gamma_0(N), \chi_{-N})$.

3 Proof of Theorem1.1

Since N is a squarefree integer, a complete set of representatives for the cusps of $\Gamma_0(N)$ is

$$\mathcal{C}_N = \{\frac{1}{a} : a|N\}.$$

Let $\frac{1}{k} \in \mathcal{C}_N$. We calculate the order of zero of $\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)$ and $\eta(\tau)\eta(N\tau)$ at $\frac{1}{k}$. First, we consider the eta-product. From (1) , the order of zero of $\eta(\tau)\eta(N\tau)$ at $\frac{1}{k}$ is

$$\nu_{\frac{1}{k}} = \frac{N + k^2}{24k}.$$

We have $\nu_{\frac{1}{k}} = \frac{N+k^2}{24k} \in \mathbf{N}$, because k devides N and 24 devides $N + 1$ and $k^2 - 1$. From this, $\eta(\tau)\eta(N\tau)$ vanishes at all cusps of $\Gamma_0(N)$. Hence we obtain $\eta(\tau)\eta(N\tau) \in \mathcal{S}_1(\Gamma_0(N), \chi_{-N})$.

Next, we consider the theta series. We put $A_1 = \begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix}$, $A_2 = \begin{pmatrix} 12 & 5 \\ 5 & \frac{N+25}{12} \end{pmatrix}$ and put $\vartheta_{A_1}(\tau) = \vartheta_{Q_1}(\tau)$, $\vartheta_{A_2}(\tau) = \vartheta_{Q_2}(\tau)$. For a cusp $\frac{1}{k}$, we take $\gamma = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in SL_2(\mathbf{Z})$. Then we have $\gamma\infty = \frac{1}{k}$. By transformation law (see [10], p.189), we have

$$\vartheta_{A_1}(\tau)|[\gamma]_1 = (\det A_1)^{-\frac{1}{2}} k^{-1}(-i) \sum_{\substack{\mathbf{m} \in (\mathbf{Z}/N\mathbf{Z})^2 \\ A_1\mathbf{m} \equiv 0 \pmod{N}}} \Phi(\mathbf{m})\vartheta(\tau; A_1, \mathbf{m}, N),$$

where

$$\vartheta(\tau; A_1, \mathbf{m}, N) = \sum_{\substack{\mathbf{x}\in(\mathbf{Z}/N\mathbf{Z})^2 \\ \mathbf{x}\equiv\mathbf{m}\ (\bmod N)}} q^{\frac{Q_1(\mathbf{x})}{N^2}},$$

$$\Phi(\mathbf{m}) = \sum_{\substack{\mathbf{g}\in(\mathbf{Z}/kN\mathbf{Z})^2 \\ \mathbf{g}\equiv 0\ (\bmod N)}} \mathbf{e}(\frac{1}{kN^2}\{\frac{1}{2}{}^t\mathbf{g}A_1\mathbf{g} + {}^t\mathbf{m}A_1\mathbf{g} + \frac{1}{2}{}^t\mathbf{m}A_1\mathbf{m}\}).$$

Hence $\vartheta_{A_1}(\tau)|[\gamma]_1$ has a q_{h_k} expansion ($q_{h_k} = q^{\frac{1}{h_k}}$)

$$\vartheta_{A_1}(\tau)|[\gamma]_1 = (\det A_1)^{-\frac{1}{2}}k^{-1}(-i) \sum_{\substack{\mathbf{m}\in(\mathbf{Z}/N\mathbf{Z})^2 \\ A_1\mathbf{m}\equiv 0\ (\bmod N)}} \Phi(\mathbf{m})q_{h_k}^{\frac{Q_1(\mathbf{m})h_k}{N^2}}.$$

Lemma

$$\min\{\frac{Q_1(\mathbf{m})h_k}{N^2} : \mathbf{m}\in\mathbf{Z}^2,\ A_1\mathbf{m}\equiv\mathbf{0}\ (\bmod N)\} \geq \frac{N+k^2}{24k}.$$
$$\min\{\frac{Q_2(\mathbf{m})h_k}{N^2} : \mathbf{m}\in\mathbf{Z}^2,\ A_2\mathbf{m}\equiv\mathbf{0}\ (\bmod N)\} \geq \frac{N+k^2}{24k}.$$

Proof. We put $\mu_1 = \mu_1(\mathbf{m}) := \frac{Q_1(\mathbf{m})h_k}{N^2}$. Then

$$6x^2 + xy + (\frac{N+1}{24}y^2 - \frac{\mu_1 N^2}{h_k}) = 0$$

has integral solutions. We regard above as a quadratic equation of x. Then its discriminant

$$\begin{aligned} D &= y^2 - 24(\frac{N+1}{24}y^2 - \frac{\mu_1 N^2}{h_k}) \\ &= N(-y^2 + 24\mu_1 k) \end{aligned}$$

is a square. Since N is squarefree , there exists $\alpha \in 2\mathbf{N}+1$, $s \in \mathbf{N}$ such that

$$-y^2 + 24\mu_1 k = N^\alpha s^2.$$

From this, we have

$$\begin{aligned} y^2 &= 24\mu_1 k - N^\alpha s^2 \\ &= k(24\mu_1 - h_k N^{\alpha-1}s^2). \end{aligned}$$

Hence there exists $\beta \in 2\mathbf{N}+1$, $t \in \mathbf{N}$ such that

$$24\mu_1 - h_k N^{\alpha-1}s^2 = k^\beta t^2.$$

Thus we obtain

$$\mu_1 = \frac{h_k N^{\alpha-1}s^2 + k^\beta t^2}{24} \geq \frac{h_k + k}{24} = \frac{N+k^2}{24k}.$$

A similar argument works for $\vartheta_{Q_2}(\tau)$. $\square$

By lemma we have

$$\frac{\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)}{\eta(\tau)\eta(N\tau)} \in \mathcal{M}_0(\Gamma). \tag{2}$$

We note that there are no non-constant modular forms of weight zero, i.e.

$$\mathcal{M}_0(\Gamma) = \mathbf{C}$$

for any congruence subgroup Γ. Hence we have $\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau) = c\eta(\tau)\eta(N\tau)$, for some $c \in \mathbf{C}$. Comparing coefficient of $q^{\frac{N+1}{24}}$, theorem1.1 follows from this.

4 Example

We give some examples for theorem1.1.

The case $\eta(\tau)\eta(71\tau)$.
Consider the two primitive binary quadratic forms with discriminant -71 :

$$Q_1 : \ 6x^2 + xy + 3y^2, \qquad Q_2 : \ 6x^2 + 5xy + 4y^2.$$

By theorem1.1 we have

$$\frac{1}{2}(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)) = \eta(\tau)\eta(71\tau) \in \mathcal{S}_1(\Gamma_0(71), \chi_{-71}).$$

Next, we show that $\eta(\tau)\eta(71\tau)$ is a linear combination of dihedral cusp forms. Let K be the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-71})$ and let H_K be the Hilbert class field of K. We consider an irreducible odd representation $\rho : Gal(H_K/\mathbf{Q}) \to GL_2(\mathbf{C})$ with Artin conductor 71. Then ρ is determined by a character $\chi : H(-71) \to \mathbf{C}^*$, namely $\rho = Ind\,{}^{Gal(H_K/\mathbf{Q})}_{Gal(H_K/K)}(\chi)$. The elements of group $H(-71) \cong C_7$ (cyclic group of order 7) are written as follows:

$$H(-71) = \begin{cases} R_0 : \ x^2 + xy + 18y^2 \\ R_1 : \ 2x^2 + xy + 9y^2 \\ R_2 : \ 4x^2 + 3xy + 5y^2 \\ R_3 : \ 3x^2 + xy + 6y^2 \\ R_4 : \ 3x^2 - xy + 6y^2 \\ R_5 : \ 4x^2 - 3xy + 5y^2 \\ R_6 : \ 2x^2 - xy + 9y^2 \end{cases}$$

where R_0 is the identity element, R_1 is a generator of group $H(-71)$ and $R_i = {R_1}^i$.

Since $Gal(H_K/\mathbf{Q}) \cong D_{14}$, the number of irreducible representation of $Gal(H_K/\mathbf{Q})$ is 3. We put $\rho_i = Ind_{Gal(H_K/K)}^{Gal(H_K/\mathbf{Q})}(\chi_i)$, $\chi_i(R_1) = \zeta_7^i$ i.e. $\rho_i(R_1) = \begin{pmatrix} \zeta_7^i & 0 \\ 0 & \zeta_7^{-i} \end{pmatrix}$ $(i = 1, 2, 3)$. For this ρ_i, there exists a normalized newform g_i in $\mathcal{S}_1(\Gamma_0(71), \chi_{-71})$ such that $L(\rho_i, s) = L(g_i, s)$. It is given by

$$g_i = \frac{1}{2} \sum_{R \in H(-71)} \chi_i(R)\ \vartheta_R(\tau). \tag{3}$$

We consider the space which is generated by $\{\vartheta_R(\tau) \mid R \in H(-71)\}$. Let φ and ψ be Dirichlet characters modulo u and v with $uv = N$ and φ is primitive and $(\varphi\psi)(-1) = -1$. Then Eisenstein series $E^{\varphi,\psi}$ is defined by

$$E_1^{\varphi,\psi} = \delta(\varphi)L(\psi, 0) + \delta(\psi)L(\varphi, 0) + 2\sum_{n=1}^{\infty} \sum_{\substack{m|n \\ m \geq 1}} \varphi(\frac{n}{m})\psi(m)q^n$$

where

$$\delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \text{ is trivial} \\ 0 & \text{if otherwise} \end{cases}.$$

From (3), we have $g_i \in < \vartheta_R(\tau) \mid R \in H(-71) >$ $(1 \leq i \leq 3)$, and we see that Eisenstein series $E_1^{1,\chi_{-71}} \in < \vartheta_R(\tau) \mid R \in H(-71) >$. More precisely, we have

$$\begin{pmatrix} E_1^{1,\chi_{-71}}/2 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & a_1 & a_2 & a_3 \\ 1 & a_2 & a_3 & a_1 \\ 1 & a_3 & a_1 & a_2 \end{pmatrix} \begin{pmatrix} \vartheta_{R_0}(\tau) \\ \vartheta_{R_1}(\tau) \\ \vartheta_{R_2}(\tau) \\ \vartheta_{R_3}(\tau) \end{pmatrix},$$

where $a_i = \zeta_7^i + {\zeta_7}^{-i}$. Then determinant of this matrix is nonzero. Hence we have $< \vartheta_R(\tau) \mid R \in H(-71) > = < E_1^{1,\chi_{-71}},\ g_1,\ g_2,\ g_3 >$. Moreover, we have $\frac{1}{2}(\vartheta_{R_i}(\tau) - \vartheta_{R_j}(\tau))$ $(i \neq j)$ is a linear combination of cusp forms which correspond to representation of dihedral type, and we note that $\frac{1}{2}(\vartheta_{R_4}(\tau) - \vartheta_{R_2}(\tau)) = \eta(\tau)\eta(71\tau)$.

The case $\eta(\tau)\eta(95\tau)$.
Consider the two primitive binary quadratic forms with discriminant -95 :

$$Q_1 :\ 6x^2 + xy + 4y^2, \qquad Q_2 :\ 6x^2 + 5xy + 5y^2.$$

By theorem1.1 we have

$$\frac{1}{2}(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)) = \eta(\tau)\eta(95\tau) \in \mathcal{S}_1(\Gamma_0(95), \chi_{-95}).$$

Next, we show that $\eta(\tau)\eta(95\tau)$ is a linear combination of dihedral cusp forms. Let K be the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-95})$ and let H_K be the Hilbert class field of K. We consider an irreducible odd representation $\rho : Gal(H_K/\mathbf{Q}) \to GL_2(\mathbf{C})$ with Artin conductor 95. Then ρ is determined by a character $\chi : H(-95) \to \mathbf{C}^*$, namely $\rho = Ind\,^{Gal(H_K/\mathbf{Q})}_{Gal(H_K/K)}(\chi)$. The elements of group $H(-95) \cong C_8$ are written as follows:

$$H(-95) = \begin{cases} R_0 : & x^2 + xy + 24y^2 \\ R_1 : & 2x^2 + xy + 12y^2 \\ R_2 : & 4x^2 + xy + 6y^2 \\ R_3 : & 3x^2 + xy + 8y^2 \\ R_4 : & 5x^2 + 5xy + 6y^2 \\ R_5 : & 3x^2 - xy + 8y^2 \\ R_6 : & 4x^2 - xy + 6y^2 \\ R_7 : & 2x^2 - xy + 12y^2 \end{cases}$$

where R_0 is the identity element, R_1 is a generator of group $H(-95)$ and $R_i = {R_1}^i$.

Since $Gal(H_K/\mathbf{Q}) \cong D_{16}$, the number of irreducible representation of $Gal(H_K/\mathbf{Q})$ is 3. We put $\rho_i = Ind\,^{Gal(H_K/\mathbf{Q})}_{Gal(H_K/K)}(\chi_i)$, $\chi_i(R_1) = \zeta_8^i$ i.e. $\rho_i(R_1) = \begin{pmatrix} \zeta_8^i & 0 \\ 0 & \zeta_8^{-i} \end{pmatrix}$ $(i = 1, 2, 3)$. For this ρ_i, there exists a normalized newform g_i in $\mathcal{S}_1(\Gamma_0(95), \chi_{-95})$ such that $L(\rho_i, s) = L(g_i, s)$. It is given by

$$g_i = \frac{1}{2} \sum_{R \in H(-95)} \chi_i(R)\ \vartheta_R(\tau). \tag{4}$$

Then we obtain

$$\frac{1}{4}(g_1 - 2g_2 + g_3) = \frac{1}{2}(\vartheta_{R_6} - \vartheta_{R_4}).$$

Now Q_1 and R_6 are equivalent over $SL_2(\mathbf{Z})$, namely there is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ such that $Q_1(x, y) = R_6(ax + by, cx + dy)$. Similaly Q_2 and R_4 are equivalent over $SL_2(\mathbf{Z})$. Therefore we have $\vartheta_{Q_1} = \vartheta_{R_6}$, $\vartheta_{Q_2} = \vartheta_{R_4}$ and

$$\frac{1}{4}(g_1 - 2g_2 + g_3) = \frac{1}{2}(\vartheta_{R_6} - \vartheta_{R_4}) = \eta(\tau)\eta(95\tau).$$

This mean that $\eta(\tau)\eta(95\tau)$ is a linear combination of cusp forms which correspond to representation of dihedral type.

Lastly, we consider the space which is generated by $\{\vartheta_R(\tau) \mid R \in H(-95)\}$. From (4), we have $g_i \in <\vartheta_R(\tau) \mid R \in H(-95)>$ $(1 \leq i \leq 3)$, and we see that Eisenstein series $E_1^{1,\chi_{-95}}$ and $E_1^{\chi_5,\chi_{-19}} \in <\vartheta_R(\tau) \mid R \in H(-95)>$. More precisely, we have

$$\begin{pmatrix} E_1^{1,\chi_{-95}}/2 \\ E_1^{\chi_5,\chi_{-19}}/2 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & -2 & 2 & -2 & 1 \\ 1 & \sqrt{2} & 0 & -\sqrt{2} & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \vartheta_{R_0}(\tau) \\ \vartheta_{R_1}(\tau) \\ \vartheta_{R_2}(\tau) \\ \vartheta_{R_3}(\tau) \\ \vartheta_{R_4}(\tau) \end{pmatrix}.$$

Then determinant of this matrix is nonzero. Hence we have $<\vartheta_R(\tau) \mid R \in H(-95)> = <E_1^{1,\chi_{-95}}, E_1^{\chi_5,\chi_{-19}}, g_1, g_2, g_3>$. Moreover, suppose that R_i and R_j are in the same genus i.e. equivalent over the p-adic integers $\mathbf{Z}_p$ for all primes p and equivalent over $\mathbf{R}$. Then we see that $\frac{1}{2}(\vartheta_{R_i}(\tau) - \vartheta_{R_j}(\tau))$ is a linear combination of cusp forms which correspond to representation of dihedral type.

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